STABILITY OF UNIVERSAL EQUIVALENCE OF GROUPS UNDER FREE CONSTRUCTIONS

ANDREW J. DUNCAN, ILYA V. KAZACHKOV, AND VLADIMIR N. REMESLENNIKOV

1. Introduction

In his important paper in [3] J. Stallings introduced a generalisation of amalgamated products of groups – called a pregroup, which is a particular kind of a partial group. He then defined the universal group $U(P)$ of a pregroup P to be a universal object (in the sense of category theory) extending the partial operations on P to group operations on $U(P)$. The universal group turned out to be a versatile and convenient generalisation of classical group constructions: HNN-extensions and amalgamated products. In this respect the following general question arises.Which properties of pregroups, or relations between pregroups, carry over to the respective universal groups? The aim of this paper is to prove that universal equivalence of pregroups extends to universal equivalence of their universal groups.

We begin by some preliminary model-theory results. We refer the reader to [2] for a detailed introduction to model theory. The main goal here is to give a criterion of universal equivalence of two models in the form that best suits our needs.

2. Preliminaries

Let $\mathcal{L}$ be a language with signature (C, F, R), and variables X, where C is a set of constants and F and R are finite sets of functions and relations respectively. In addition each element f of F is associated to a non-negative integer n_f, and similarly for R.

An $\mathcal{L}$-structure $\mathcal{M}$ is a 4-tuple:

- a non-empty set M;
- a function $f_{\mathcal{M}} : M^{n_f} \to M$ for each $f \in F$;
- a set $r_{\mathcal{M}} \subseteq M^{n_r}$ for each $r \in R$;
- an element $c_{\mathcal{M}}$ for each element $c \in C$.

If a is an element of C, F or R we refer to $a_{\mathcal{M}}$ as the interpretation of a in $\mathcal{M}$. The subscript $\mathcal{M}$ is omitted where no ambiguity arises.

We use the language $\mathcal{L}$ to write formulas describing the properties of $\mathcal{L}$-structures. Roughly speaking formulas are constructed inductively starting from constant symbols from C and variable symbols $v_1, \dots, v_n, \dots$, using the Boolean connectives, relations from R, functions from F and the equality symbol '='.

More precisely, the set of $\mathcal{L}$-*terms* is the smallest set T such that:

- $c \in T$ for each constant symbol $c \in C$;
- each variable symbol $v_i \in T$;
- if $t_1, \dots, t_n \in T$ and $f \in F$ then $f(t_1, \dots, t_{n_f}) \in T$.

Research supported by EPSRC grant EP/D065275.

We say that Φ is an *atomic $\mathcal{L}$-formula* if Φ is either

- $t_1 = t_2$, where t_1 and t_2 are terms or,
- $r(t_1, \dots, t_{n_r})$, where $r \in R$ and $t_1, \dots, t_{n_r}$ are terms.

The set of *$\mathcal{L}$-formulas* is the smallest set W containing atomic formulas and such that

- if $\Phi \in W$ then $\neg\Phi \in W$;
- if Φ and Ψ are in W then $\Phi \wedge \Psi$ and $\Phi \vee \Psi$ are in W and
- if Φ is in W then $\exists v_i \Phi$ and $\forall v_i \Phi$ are in W.

It is often useful in practice to observe that, as

$$\Phi \vee \Psi \equiv \neg(\neg\Phi \wedge \neg\Psi)$$

and

$$\forall v_i \Phi \equiv \neg\exists v_i(\neg\Phi)$$

we can construct all formulas (up to logical equivalence $\equiv$) without using $\vee$ or $\forall$.

To make induction arguments precise we shall define, for any term or formula s of $\mathcal{L}$, the *level* $l(s)$ and the *constants* $C(s)$ of s. If s is a formula we shall also define the *degree* $d(s)$ of s. To begin with if t is a term and $t = x$ or $t = c$, where x is a variable and c a constant, then $l(t) = 0$ and

$$C(t) = \begin{cases} \emptyset, & \text{if } t = x \\ c, & \text{if } t = c \end{cases}.$$

If $t = f(t_1, \dots, t_n)$, where $n = n_f$ and the t_i are terms then $l(t) = \max\{l(t_1), \dots, l(t_n)\} + 1$, and $C(t) = \cup_{i=1}^n C(t_i)$.

If a is an atomic formula of the form $t_1 = t_2$, for terms t_1 and t_2, then we define $l(a) = \max\{l(t_1), l(t_2)\}$ and $C(t) = C(t_1) \cup C(t_2)$. If r is an n-ary relation and $a = r(t_1, \dots, t_n)$ then set $l(a) = \max\{l(t_1), \dots, l(t_n)\}$ and $C(a) = \cup_{i=1}^n C(t_i)$. The degree of an atomic formula a is defined to be $d(a) = 0$.

If $\Phi = \neg\Psi$ or $\Phi = \exists x\Psi$ or $\forall x\Psi$ then we define $l(\Phi) = l(\Psi)$, $d(\Phi) = d(\Psi) + 1$ and $C(\Phi) = C(\Psi)$. If $\Phi = \Phi_1 \wedge \Phi_2$ or $\Phi_1 \vee \Phi_2$ then we define $l(\Phi) = \max\{l(\Phi_1), l(\Phi_2)\}$, $d(\Phi) = d(\Phi_1) + d(\Phi_2)$ and $C(\Phi) = C(\Phi_1) \cup C(\Phi_2)$.

We say that a variable v occurs freely in a formula Φ if it is not inside a $\exists v$ or a $\forall v$ quantifier, otherwise v is said to be bound. A formula is called a *sentence* or *closed* if it has no free variables.

Let Φ be a formula with free variables from $v = (v_1, \dots, v_m)$ and let $\bar{a} = (a_1, \dots, a_m) \in M^m$. We inductively define when Φ *holds* on $\bar{a}$ in a $\mathcal{L}$-structure $\mathcal{M}$ ($\Phi(\bar{a})$ is *true* in $\mathcal{M}$ or $\mathcal{M}$ *satisfies* $\Phi(\bar{a})$), write $\mathcal{M} \models \Phi(\bar{a})$.

- if Φ is $t_1 = t_2$, then $\mathcal{M} \models \Phi(\bar{a})$ if $t_1(\bar{a}) = t_2(\bar{a})$;
- if $\Phi = r(t_1, \dots, t_{n_r})$, then $\mathcal{M} \models \Phi(\bar{a})$ if $r(t_1(\bar{a}), \dots, t_{n_r}(\bar{a})) \in r_{\mathcal{M}}$;
- if $\Phi = \neg\Psi$ then $\mathcal{M} \models \Phi(\bar{a})$ if $\mathcal{M} \not\models \Psi(\bar{a})$;
- if $\Phi = \Psi_1 \wedge \Psi_2$ then $\mathcal{M} \models \Phi(\bar{a})$ if $\mathcal{M} \models \Psi_1(\bar{a})$ and $\mathcal{M} \models \Psi_2(\bar{a})$;
- if $\Phi = \Psi_1 \vee \Psi_2$ then $\mathcal{M} \models \Phi(\bar{a})$ if $\mathcal{M} \models \Psi_1(\bar{a})$ or $\mathcal{M} \models \Psi_2(\bar{a})$;
- if $\Phi = \exists v_{m+1}\Psi(\bar{v}, v_{m+1})$, then $\mathcal{M} \models \Phi$ if there exists $b \in M$ such that $\mathcal{M} \models \Psi(\bar{a}, b)$;
- if $\Phi = \forall v_{m+1}\Psi(\bar{v}, v_{m+1})$, then $\mathcal{M} \models \Phi$ if for all $b \in M$ one has $\mathcal{M} \models \Psi(\bar{a}, b)$

A set of sentences is called a *theory*. We say that $\mathcal{M}$ is a *model* of a theory T if $\mathcal{M} \models \Phi$ for all $\Phi \in T$. For an $\mathcal{L}$-structure $\mathcal{M}$ we denote by $\mathtt{Th}(\mathcal{M})$ the collection of

all sentences that are satisfied by $\mathcal{M}$, $\mathtt{Th}(\mathcal{M})$ is called the *full* or *elementary theory* of $\mathcal{M}$.

Every formula Φ of $\mathcal{L}$ with free variables $\bar{v} = (v_1, \dots, v_k)$ is logically equivalent to a formula of the type

$$Q_1x_1Q_2x_2 \dots Q_nx_n\Psi(\bar{x}, \bar{v}),$$

where $Q_i \in \{\forall, \exists\}$, and $\Psi(\bar{x}, \bar{v})$ is a boolean combination of atomic formulas in variables from $\bar{v} \cup \bar{x}$. This form is called the *prenex normal form* of a formula Φ.

A sentence Φ is called *universal* (*existential*) if Φ is equivalent to a formula of the form

$$Q_1x_1Q_2x_2 \dots Q_nx_n\Psi(\bar{x}),$$

where $Q_i = \forall$ ($Q_i = \exists$) for all i, and $\Psi(\bar{x})$ is a boolean combination of atomic formulas in the indicated variables. The collection of all universal (existential) sentences that are satisfied by an $\mathcal{L}$-structure $\mathcal{M}$ is called the *universal* (*existential*) theory of $\mathcal{M}$, we denote it by $\mathtt{Th}_\forall(\mathcal{M})$ ($\mathtt{Th}_\exists(\mathcal{M})$). If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures and $\mathtt{Th}_\forall(\mathcal{M}) = \mathtt{Th}_\forall(\mathcal{N})$ we say that $\mathcal{M}$ and $\mathcal{N}$ are *universally equivalent* and write $\mathcal{M} \equiv_\forall \mathcal{N}$. *existential equivalence* is defined similarly and we write $\mathcal{M} \equiv_\exists \mathcal{N}$ if $\mathcal{M}$ and $\mathcal{N}$ are existentially equivalent.

Let A be a set of sentences of $\mathcal{L}$ and let $\mathcal{M}$ and $\mathcal{N}$ be models of A with underlying sets M and N respectively. For subsets S and T of M and N respectively we say that a map $\phi : S \to T$ is an $\mathcal{L}$*-morphism* if the following conditions hold.

(i) If $c \in C \cap S$ then $c \in T$ and $\phi(c) = c$.

(ii) If f is an n-ary function (i.e. $n_f = n$) in F and $f(s_1, \dots, s_n) \in S$, for some n-tuple $(s_1, \dots, s_k)$ of elements of S, then

$$\phi(f(s_1, \dots, s_n)) = f(\phi(s_1), \dots, \phi(s_n)) \in T.$$

(iii) If r is an n-ary relation in R and $(s_1, \dots, s_n) \in r$, for some n-tuple $(s_1, \dots, s_n)$ of elements of S, then $(\phi(s_1), \dots, \phi(s_n)) \in r$.

If $\phi : S \to T$ is a bijective $\mathcal{L}$-morphism such that ϕ^{-1} is an $\mathcal{L}$-morphism from T to S then we say that ϕ is an $\mathcal{L}$*-isomorphism* and that S and T are $\mathcal{L}$*-isomorphic* or $S \cong_\mathcal{L} T$. $\mathcal{L}$-isomorphism defines an equivalence relation on the subsets of a model $\mathcal{M}$ and we denote by $[S]$ the equivalence class of S.

Now we restrict attention to finite subsets of models. We denote by $\mathcal{F}_\mathcal{L}(\mathcal{M}) = \mathcal{F}(\mathcal{M})$ the set of $\mathcal{L}$-isomorphism equivalence classes of finite subsets of M. We say that models $\mathcal{M}$ and $\mathcal{N}$ have *equivalent* $\mathcal{L}$-isomorphism classes of finite subsets, and write $\mathcal{F}(\mathcal{M}) \equiv \mathcal{F}(\mathcal{N})$, if there exists a bijection $\theta : \mathcal{F}(\mathcal{M}) \to \mathcal{F}(\mathcal{N})$ such that, for all finite subsets $S \subseteq M$, if $\theta([S]) = [T]$ then there exists an $\mathcal{L}$-isomorphism $\phi(S) \to T'$, for some $T' \in [T]$ (hence for all $T' \in [T]$).

Lemma 2.1. *$\mathcal{F}(\mathcal{M}) \equiv \mathcal{F}(\mathcal{N})$ if and only if, for all finite subsets $S \subseteq M$, there exists a subset $T \subseteq N$ such that $S \cong_\mathcal{L} T$.*

Proof. If $\mathcal{F}(\mathcal{M}) \equiv \mathcal{F}(\mathcal{N})$ and S is a finite subset of M then, by definition, S is $\mathcal{L}$-isomorphic to some finite subset of N. Conversely, suppose every finite subset of M is $\mathcal{L}$-isomorphic to a finite subset of N. For each isomorphism class U of finite subsets of M choose a representative S_U, so $U = [S_U]$. Similarly choose a representative T_V for each isomorphism class of finite subsets of N. Consider an isomorphism class $U \in \mathcal{F}(\mathcal{M})$. S_U is $\mathcal{L}$-isomorphic to T for some finite subset of N. Let T' be the chosen representative of $[T]$. Then $S_U \cong_\mathcal{L} T'$. Define $\theta(U) = [T']$. Then θ is a well-defined map from $\mathcal{F}(\mathcal{M})$ to $\mathcal{F}(\mathcal{N})$ and straightforward verification

shows that θ is a bijection. By construction, if $\theta(U) = V$ then S_U is $\mathcal{L}$-isomorphic to the representative T' of V, so the same goes of any element $S \in U$. Hence $\mathcal{F}(\mathcal{M}) \equiv \mathcal{F}(\mathcal{N})$. □

If $\mathcal{M}$ and $\mathcal{N}$ are models of A then it is easy to see that $\mathcal{M} \equiv_\exists \mathcal{N}$ if and only if $\mathcal{M} \equiv_\forall \mathcal{N}$. The following proposition gives a further characterisation of this property, in certain cases.

Proposition 2.2. *Assume that $\mathcal{L}$ has signature (C, F, R) where either*
- (i) *C is finite or*
- (ii) *R contains a relation δ_C and, for each $c \in C$, A contains axioms*
  - *(a) $c \in \delta_C$ and*
  - *(b) $\forall x(x \notin \delta_C \implies x \neq c)$.*

Let $\mathcal{M}$ and $\mathcal{N}$ be models of A. Then $\mathcal{M} \equiv_\exists \mathcal{N}$ if and only if $\mathcal{F}(\mathcal{M}) \equiv \mathcal{F}(\mathcal{N})$.

Proof. Assume first that $\mathcal{M} \equiv_\exists \mathcal{N}$. Let $S = \{m_1, \ldots, m_k\}$ be a finite subset of $\mathcal{M}$. Define the formula

$$\Phi_1 = \bigwedge_{1 \le i < j \le k} x_i \neq x_j.$$

Φ_1 will enable us to identify k distinct elements of M or N; in fact $\mathcal{M} \vDash \Phi_1[m_1, \ldots, m_k]$ so $\mathcal{M} \vDash \exists x_1, \ldots, x_k \Phi_1$.

Now let $S \cap C = \{m_{i_1}, \ldots, m_{i_s}\}$, say $m_{i_j} = c_j \in C$ and let $\{1, \ldots, k\} \backslash \{i_1, \ldots, i_s\} = \{j_1, \ldots, j_t\}$. Define

$$\Phi_2 = \left(\bigwedge_{r=1}^{s} x_{i_r} = c_r \right) \wedge \left(\bigwedge_{c \in C} \bigwedge_{r=1}^{t} x_{j_r} \neq c \right),$$

if C is finite and

$$\Phi_2 = \left(\bigwedge_{r=1}^{s} x_{i_r} = c_r \right) \wedge \left(\bigwedge_{r=1}^{t} x_{j_r} \notin \delta_C \right),$$

otherwise. By construction $\mathcal{M} \vDash \Phi_2[m_1, \ldots, m_k]$ and Φ_2 allows us to identify $C \cap S$ and a corresponding subset of N.

Write $I_k = \{1, \ldots, k\}$. Let $f \in F$ be an n-ary function, for some $n \geq 1$. Let

$$S_{f,0} = \{(i_1, \ldots, i_n) \in I_k^n | f(m_{i_1}, \ldots, m_{i_n}) \in S \backslash C\}$$

and

$$S_{f,1} = \{(i_1, \ldots, i_n) \in I_k^n | f(m_{i_1}, \ldots, m_{i_n}) \in C\}.$$

For each $(i_1, \ldots, i_n) \in S_{f,1}$ define $s = s(i_1, \ldots, i_n)$ to be the integer in I_k such that $f(m_{i_1}, \ldots, m_{i_n}) = m_s$. Define

$$\Phi_{f,0} = \bigwedge_{(i_1, \ldots, i_n) \in S_{f,0}} f(x_{i_1}, \ldots, x_{i_n}) = f(m_{i_1}, \ldots, m_{i_n})$$

and

$$\Phi_{f,1} = \bigwedge_{(i_1, \ldots, i_n) \in S_{f,1}} f(x_{i_1}, \ldots, x_{i_n}) = m_{s(i_1, \ldots, i_n)}.$$

Define $\Phi_f = \Phi_{f,0} \wedge \Phi_{f,1}$. Then $\mathcal{M} \vDash \Phi_f[m_1, \ldots, m_k]$.

Let $r \in R$ be an n-ary relation, for some $n \geq 1$, and let

$$S_r = \{(i_1, \ldots, i_n) \in I_k | (m_{i_1}, \ldots, m_{i_n}) \in r\}.$$

Define

$$\Phi_r = \left(\bigwedge_{(i_1,\dots,i_n)\in S_r} r(x_{i_1},\dots,x_{i_n}) \right) \wedge \left(\bigwedge_{(i_1,\dots,i_n)\notin S_r} \neg r(x_{i_1},\dots,x_{i_n}) \right)$$

Then $\mathcal{M} \vDash \Phi_r[m_1,\dots,m_k]$.

Finally define $\Phi = \Phi_1 \wedge \Phi_2 \wedge \bigwedge_{f\in F} \Phi_f \wedge \bigwedge_{r\in R} \Phi_r$. Then $\mathcal{M} \vDash \Phi[m_1,\dots,m_k]$ so $\mathcal{M} \vDash \exists x_1,\dots,x_k\Phi$. Therefore $\mathcal{N} \vDash \exists x_1,\dots,x_k\Phi$ and there exist $n_1,\dots,n_k \in N$ such that $\mathcal{N} \vDash \Phi[n_1,\dots,n_k]$.

Set $T = \{n_1,\dots,n_k\}$ and define $\phi : S \to T$ by $\phi(m_i) = n_i$, $i = 1,\dots,k$. By definition ϕ is an $\mathcal{L}$-morphism and is a bijection of S and T. Moreover ϕ^{-1} is, by construction of Φ, an $\mathcal{L}$-morphism. Hence $S \cong_{\mathcal{L}} T$ and it follows from Lemma 2.1 that $\mathcal{F}(\mathcal{M}) \equiv \mathcal{F}(\mathcal{N})$.

Now suppose that $\mathcal{F}(\mathcal{M}) \equiv \mathcal{F}(\mathcal{N})$. Write F_n for the set of n-ary functions of F. Since F is finite we may assume that F is the union of F_n, for n from 1 to K, for some $K \in \mathbb{N}$. Given a finite subset S of M we define the following sequence of subsets. Set $S_0 = S$ and having defined S_i set

$$S_{i+1} = S_i \cup \bigcup_{n=1,\dots,K} \bigcup_{f\in F_n} \{f(m_1,\dots,m_n) | m_j \in S_i, j = 1,\dots,n\}.$$

Now choose $T_l \subseteq N$ such that there is an $\mathcal{L}$-isomorphism ϕ_l from S_l to T_l, for all $l \geq 0$.

Consider a term t of level l with variables among $x_1,\dots,x_k$ and a k-tuple $a_1,\dots,a_k$ of elements of M and set $S = \{a_1,\dots,a_k\} \cup C(t)$. We claim that $t(a_1,\dots,a_k) \in S_l$. To see this note that it holds when $l = 0$, since in this case $t(a_1,\dots,a_k) \in S$. Suppose then that t has level l and that the claim holds for at all levels below l. Then $t = f(t_1,\dots,t_m)$, where $l(t_i) < l$. By assumption $t_i(a_1,\dots,t_k) \in S_{l-1}$ and so by definition $t(a_1,\dots,a_k) = f(t_1(a_1,\dots,t_k),\dots,t_m(a_1,\dots,t_k)) \in S_l$; and the claim holds for all l by induction.

Let Φ be a quantifier free formula with variables among $x_1,\dots,x_m$ and let $\Psi = \exists x_1,\dots,x_m\Phi$. We wish to show that $\mathcal{M} \vDash \Psi$ if and only if $\mathcal{N} \vDash \Psi$. To do this we shall proceed as follows. Suppose Φ has level l and let $a_1,\dots,a_m \in M$. Let $S = S(\Phi) = \{a_1,\dots,a_m\} \cup C(\Phi)$, where $C(\Phi)$ is the set of constants of Φ, and define $S_0, S_1,\dots$ and $\phi_1,\phi_2,\dots$ as above. We shall prove that

$$\mathcal{M} \vDash \Phi(a_1,\dots,a_m) \text{ if and only if } \mathcal{N} \vDash \Phi(\phi_l(a_1),\dots,\phi_l(a_m)), \tag{1}$$

and the result will follow immediately. We use induction on (d,l), where d is the degree and l the level of Φ.

Assume that (1) holds whenever Φ has level at most l and degree 0. Suppose now that Φ has level $l+1$ and degree 0. In this case Φ is of the form $t_1 = t_2$, or of the form $r(t_1,\dots,t_m)$, where $r \in R$ and the t_i are terms. Since Φ has level $l+1$ at least one of the t_i has level $l+1$ and none have level greater than $l+1$. Hence $t_i(a_1,\dots,a_k) \in S_{l+1}$, for all i. Set $b_i = \phi_{l+1}(a_i)$, $i = 1,\dots,k$. Then, as ϕ_{l+1} is an isomorphism with domain S_{l+1}, we have $\phi_{l+1}(t_i(a_1,\dots,a_k)) = t_i(b_1,\dots,b_k)$, for all i. Furthermore $t_1(a_1,\dots,a_k) = t_2(a_1,\dots,a_k)$ if and only if $t_1(b_1,\dots,b_k) = t_2(b_1,\dots,b_k)$ and $(t_1(a_1,\dots,a_k),\dots,t_m(a_1,\dots,a_k)) \in r$ if and only if $(t_1(b_1,\dots,b_k),\dots,t_m(b_1,\dots,b_k)) \in r$. Hence $\mathcal{M} \vDash \Phi(a_1,\dots a_k)$ if and only if $\mathcal{N} \vDash \Phi(b_1,\dots,b_k)$. Therefore the result holds for Φ of level $l+1$ and degree 0.

Note that this argument also goes through in the case $(d,l)=(0,0)$ so by induction (1) holds for formulae Φ of level l and degree 0, for all non-negative integers l.

Now let d and l be non-negative integers and assume that (1) holds for formulae Φ of degree d_1 and level l_1 where either (i) $d_1 \le d$ and $l_1 = l$ or (ii) $l_1 < l$. Suppose then that Φ has level l and degree $d+1$. Then either $\Phi = \neg\Phi_1$ or $\Phi = \Phi_1 \wedge \Phi_2$, where Φ_1 and Φ_2 have degree at most d and level at most l. If $\Phi = \neg\Phi_1$ then $\mathcal{M} \vDash \Phi_1(a_1,\dots,a_m)$ if and only if $\mathcal{N} \vDash \Phi_1(a_1,\dots,a_m)$, so the same holds with Φ in place of Φ_1. If $\Phi = \Phi_1 \wedge \Phi_2$ then $\mathcal{M} \vDash \Phi(a_1,\dots,a_m)$ if and only if $\mathcal{M} \vDash \Phi_i(a_1,\dots,a_m)$, for $i=1$ and 2, if and only if $\mathcal{N} \vDash \Phi_i(a_1,\dots,a_m)$, for $i=1$ and 2, if and only if $\mathcal{N} \vDash \Phi(a_1,\dots,a_m)$. It follows that (1) holds for Φ of level l and any degree $d+1$; hence by induction for all (d,l). □

We call an expression of the form $t_1 = t_2$, where t_1 and t_2 are terms, an *equation*. A set S of equations such that every element of S has variables among $x_1,\dots,x_m$ is called a *system of equations* in m variables. Let S be a system of equations in m variables and let $\mathcal{M}$ be a model of $\mathcal{L}$. We say that $(a_1,\dots,a_m) \in M^m$ is a *solution* of S in $\mathcal{M}$ if $\mathcal{M} \vDash s(a_1,\dots,a_m)$, for all $s \in S$. The *variety* defined by S over $\mathcal{M}$ is the set $V_{\mathcal{M}}(S) = \{(a_1,\dots,a_m) \in M^m : (a_1,\dots,a_m) \text{ is a solution of } S\}$. We say that a model $\mathcal{M}$ of $\mathcal{L}$ is *equationally Noetherian* if every system S of equations contains a finite subset S_0 such that $V_{\mathcal{M}}(S_0) = V_{\mathcal{M}}(S)$. As in [1] we have the following lemma.

Lemma 2.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. Then,*

- (i) *if $\mathcal{M}$ is equationally Noetherian and $\mathtt{Th}_\exists(\mathcal{N}) \subseteq \mathtt{Th}_\exists(\mathcal{M})$, then $\mathcal{N}$ is equationally Noetherian;*
- (ii) *if $\mathcal{M}$ and $\mathcal{N}$ are universally equivalent, $\mathcal{M}$ is equationally Noetherian if and only if $\mathcal{N}$ is equationally Noetherian.*

Proof. Suppose that $\mathcal{M}$ is equationally Noetherian and that S is a system of equations in m variables. Choose a subset $S_0 \subseteq S$ such that $V_{\mathcal{M}}(S) = V_{\mathcal{M}}(S_0)$. Let $S_0 = \{s_1,\dots,s_r\}$ and for each $s \in S$ let Φ_s be the sentence $\forall x_1,\dots,x_m(s_1 \wedge \cdots \wedge s_r \to s)$. Since $V_{\mathcal{M}}(S_0) = V_{\mathcal{M}}(S)$ we have $\mathcal{M} \vDash \Phi_s$ and therefore, since under the assumptions of any of the two statements above $\mathtt{Th}_\exists(\mathcal{N}) \subseteq \mathtt{Th}_\exists(\mathcal{M})$, we have $\mathcal{N} \vDash \Phi_s$, for all $s \in S$. As $S_0 \subseteq S$ it follows that $V_{\mathcal{N}}(S) \subseteq V_{\mathcal{N}}(S_0)$. If $(b_1,\dots,b_m) \in V_{\mathcal{N}}(S_0)$ then, as $\mathcal{N} \vDash \Phi_s$, we have $(b_1,\dots,b_m) \in V_{\mathcal{N}}(S)$, so $V_{\mathcal{N}}(S_0) = V_{\mathcal{N}}(S)$. □

3. Groups and Pregroups

The language of pregroups $\mathcal{L}^{\mathrm{pre}}$ has signature (C,F,R) where C consists of a single element 1, F consists of a unary function symbol $^{-1}$ and R consists of a binary relation D and a ternary relation M. (The usual definition of a pregroup involves a product function defined on a subset $D \subset P \times P$. Our description of language does not allow F to contain partially defined functions, so we use the relation M instead of this product. We keep the relation D for compatibility with the usual definition.) A pregroup is a model P of $\mathcal{L}^{\mathrm{pre}}$ satisfying the following axioms.

- (i) $\forall x,y,z((x,y,z) \in M \to (x,y) \in D)$.
- (ii) $\forall x,y((x,y) \in D \to \exists z((x,y,z) \in M))$.
- (iii) $\forall w,x,y,z((w,x,y) \in M \wedge (w,x,z) \in M \to y = z)$.
- (iv) $\forall x((x,1,x) \in M \wedge (1,x,x) \in M)$.

(v) $\forall x((x, x^{-1}, 1) \in M \wedge (x^{-1}, x, 1) \in M)$.
(vi) $\forall x, y, z((x, y, z) \in M \to (y^{-1}, x^{-1}, z^{-1}) \in M)$.
(vii) $\forall a, b, c, r, s, x((a, b, r) \in M \wedge (b, c, s) \in M \to ((a, s, x) \in M \leftrightarrow (r, c, x) \in M))$.
(viii) $\forall a, b, c, d, x, y, z((a, b, x) \in M \wedge (b, c, y) \in M \wedge (c, d, z) \in M \to \exists r, s((a, y, r) \in M \vee (y, d, s) \in M))$.

A pregroup homomorphism is a morphism of $\mathcal{L}^{\text{pre}}$-structures and a subpregroup is an $\mathcal{L}^{\text{pre}}$-substructure of an $\mathcal{L}^{\text{pre}}$-structure. Thus K is a subpregroup of P if and only if K is a pregroup, $K \subseteq P$, $1_K = 1_P$, $D_K = D_P \cap (K \times K)$ and $M_K = M_P \cap (K \times K \times K)$ (from which it follows that the operation of inversion in P extends that in K).

We wish, as in [1] for the group case, to consider pregroups which contain designated copies of some fixed pregroups (or some of their subsets). To this end we make the following definition.

Definition 3.1. Let M be an $\mathcal{L}$-structure and N a subset of M. The *diagram* of N is the set of all closed atomic formulas, and their negations, which hold in N.

Now let $\mathcal{S}'$ be a fixed multiset of pregroups and, for each $L \in \mathcal{S}'$, let K_L be a subset of L containing 1_L. Let $\mathcal{S}$ be the set $\{K_L | L \in \mathcal{S}'\}$. We define the language of $\mathcal{S}$-pregroups $\mathcal{L}^{\text{pre}}_{\mathcal{S}}$ to be the extension of $\mathcal{L}^{\text{pre}}$ with signature identical to $\mathcal{L}^{\text{pre}}$ except that $C = \cup_{K \in \mathcal{S}}\{d^K_k | k \in K\}$ and R contains a unary relation $\delta_{\mathcal{S}}$. A *K-pregroup* is a model P of $\mathcal{L}^{\text{pre}}_{\mathcal{S}}$ satisfying the axioms for a pregroup all the formulas of the diagram of K, for all $K \in \mathcal{S}$, and the additional axioms
(ix) $d^K_k \in \delta_{\mathcal{S}}$, for all $k \in K$, for all $K \in \mathcal{S}$, and
(x) $\forall x(x \notin \delta_{\mathcal{S}} \to x \neq d_k)$, for all $k \in K$, for all $K \in \mathcal{S}$.

(There is one axiom of type (ix) and one of type (x) for each $k \in K$ and $K \in \mathcal{S}$.) A $\mathcal{S}$-pregroup homomorphism is a morphism of $\mathcal{L}^{\text{pre}}_{\mathcal{S}}$-structures and a $\mathcal{S}$-subpregroup is an $\mathcal{L}^{\text{pre}}_{\mathcal{S}}$-substructure of an $\mathcal{L}^{\text{pre}}_{\mathcal{S}}$-structure. A $\mathcal{S}$-pregroup is finitely generated if it is finitely generated as a model of $\mathcal{L}^{\text{pre}}_{\mathcal{S}}$. If $\mathcal{S}$ consists of a single element K we call an $\mathcal{S}$-pregroup a K-pregroup and write $\mathcal{L}^{\text{pre}}_K$ instead of $\mathcal{L}^{\text{pre}}_{\mathcal{S}}$.

Lemma 3.2. *Let P be a pregroup and $a, b, c \in P$. If $(a, b, c) \in M$ then (c, b^{-1}, a) and $(c^{-1}, a, b^{-1}) \in M$.*

Proof. We have (a, b, c) and $(b, b^{-1}, 1) \in M$ and, as also $(a, 1, a) \in M$, axiom (vii) implies $(c, b^{-1}, a) \in M$. Repeating this argument starting with $(c^{-1}, c, 1)$, (c, b^{-1}, a) and $(1, b^{-1}, b)$ we see that $(c^{-1}, a, b^{-1}) \in M$. $\square$

Let $\mathcal{S}'$ be a fixed multiset of groups and, for each $G \in \mathcal{S}'$, let K_G be a subset of G containing 1_G. Let $\mathcal{S}$ be the set $\{K_G | G \in \mathcal{S}'\}$. The language of $\mathcal{S}$-groups is defined to be the language $\mathcal{L}_{\mathcal{S}}$ with signature (C, F, R), where $C = \cup_{K \in \mathcal{S}}\{d^K_k | k \in K\}$, F consists of a binary function symbol $\cdot$ and a unary function symbol $^{-1}$ and R consists of a unary relation symbol $\delta_{\mathcal{S}}$. Then an $\mathcal{S}$-group H is a model of $\mathcal{L}_{\mathcal{S}}$ satisfying the usual group axioms with respect to $\cdot$ as multiplication and $^{-1}$ as inverse in H, as well as all the formulas of the diagram of K, for all $K \in \mathcal{S}$, and the additional axioms
(a) $d^K_k \in \delta_{\mathcal{S}}$, for all $k \in K$, $K \in \mathcal{S}$, and
(b) $\forall x(x \notin \delta_{\Sigma} \implies x \neq d^K_k)$, for all $k \in K$, $K \in \mathcal{S}$.

The class of all $\mathcal{S}$-groups together with the naturally defined $\mathcal{S}$-morphisms forms a category.

If $\mathcal{S}$ consists of a single element K then we refer to K-groups instead of $\mathcal{S}$-groups and write $\mathcal{L}_K$ instead of $\mathcal{L}_\mathcal{S}$. In this case, if $K = G$ we recover the definition of G-group in [1]. Further, if $G = K = 1$ then we drop the predicate $\delta_\mathcal{S}$ from the language and we have the standard language $\mathcal{L}$ of groups. Note that, if G is a group, a G-group H is equationally Noetherian in the sense defined in the previous section if and only if it is G-equationally Noetherian in the sense of [1].

Notions of universal equivalence, elementary equivalence and equivalence of finite subsets for $\mathcal{S}$-groups are defined with respect to the language $\mathcal{L}_\mathcal{S}$; as are substructures and extensions of $\mathcal{S}$-groups. A $\mathcal{S}$-group H is locally $\mathcal{S}$-discriminated by a $\mathcal{S}$-group N if, given a finite subset $F = \{h_1, \dots, h_k\}$ of H there is a $\mathcal{S}$-homomorphism (i.e. $\mathcal{L}_\mathcal{S}$-morphism) from H to N which is injective on F. A $\mathcal{S}$-group H is said to be *finitely generated* if there exists a finite subset F of H such that H is generated by $F \cup \cup_{K\in\mathcal{S}} K$. (Thus a finitely generated $\mathcal{S}$-group is a finitely generated $\mathcal{L}_\mathcal{S}$-model.) If P is any property then a $\mathcal{S}$-group H is said to be locally P if every non-trivial finitely generated $\mathcal{S}$-subgroup of H has property P. The following theorem is proved in [1].

Theorem 3.3 ([1]). *Let G be a group and H and K be G-groups one of which is G-equationally Noetherian. Then H is locally G-discriminated by K and K is locally G-discriminated by G if and only if K and H are universally equivalent (with respect to $\mathcal{L}_G$).*

If a, b are elements of pregroup P and $(a, b) \in D_P$ we write ab for the unique element c such that $(a, b, c) \in M$. Following Stallings [3] we define a *word* of *length* k over a pregroup P to be a finite sequence $(c_1, \dots, c_k)$ of elements of P. If $(c_i, c_{i+1}) \in D$ then $c_ic_{i+1} \in P$ and the word $(c_1, \dots, c_{i-1}, c_ic_{i+1}, c_{i+1}, \dots, c_k)$ is said to be a *reduction* of $(c_1, \dots, c_k)$. The word $(c_1, \dots, c_k)$ is said to be *reduced* if $(c_i, c_{i+1}) \notin D$, for $i = 1, \dots, k-1$.

Let $\mathbf{c} = (c_1, \dots, c_k)$ and $\mathbf{a} = (a_1, \dots, a_{k-1})$ be words such that $(c_1, a_1) \in D$, (a_{i-1}^{-1}, c_i) and $(a_{i-1}^{-1}c_i, a_i)$ are in D, for $i = 1, \dots, k-1$, and $(a_{k-1}, c_k) \in D$. Then the *interleaving* $\mathbf{c} * \mathbf{a}$ of $\mathbf{c}$ and $\mathbf{a}$ is the word $(d_1, \dots, d_k)$ given by $d_1 = c_1a_1$, $d_i = a_{i-1}^{-1}c_ia_i$, for $i = 1, \dots, k-1$, and $d_k = a_{k-1}c_k$. We define a relation $\approx$ on the set of words by $\mathbf{c} \approx \mathbf{d}$ if and only if $\mathbf{d} = \mathbf{c} * \mathbf{a}$, for some word $\mathbf{a}$. As shown in [3] if $\mathbf{c}$ is reduced then so is $\mathbf{c} * \mathbf{a}$ and the relation $\approx$ is an equivalence relation on the set of reduced words over P. The *universal group* $U(P)$ of the pregroup P is the set of equivalence classes of reduced words: the group operation being concatenation of words followed by reduction to a reduced word. As P embeds in $U(P)$ then, if P is a K-pregroup it follows that $U(P)$ is a K-group. A group G may be regarded as a pregroup: with $D = G \times G$ and M the multiplication table of G. It is shown in [3] that $U(P)$ is universal in the sense that, given a group H and a pregroup morphism θ from P to H, there is a unique extension of θ to a group homomorphism from $U(P)$ to H.

Lemma 3.4. *Let P be a pregroup and let $(c_1, \dots, c_m)$ and $(d_1, \dots, d_n)$ be words. Then $(c_1, \dots, c_m) \approx (d_1, \dots, d_n)$ if and only if $m = n$ and*

$$(d_{r-1}^{-1} \cdots d_1^{-1}c_1 \cdots c_{r-1}, c_r) \in D_P \text{ and } (d_r^{-1}, d_{r-1}^{-1} \cdots d_1^{-1}c_1 \cdots c_r) \in D_P,$$

$r = 1, \dots m$, and $d_m^{-1} \cdots d_1^{-1}c_1 \cdots c_m = 1$.

Proof. Write $D = D_P$. Suppose first that $(c_1, \dots, c_m) \approx (d_1, \dots, d_n)$. Then by definition $m = n$ and there exists an interleaving $(c_1, \dots, c_m) * (a_1, \dots, a_{m-1}) =$

$(d_1,\dots,d_m)$, for some $a_i \in P$. Then (by definition again) with $a_0 = a_m = 1$ we have (a_{i-1}, c_i) and $(a_{i-1}, c_i a_i)$ in D and $d_i = a^{i-1} c_i a_i$. Thus $(c_1, a_1) \in D$ and $d_1 = c_1 a_1$. Lemma 3.2 implies that $(d_1^{-1}, c_1) \in D$ and $d_1^{-1} c_1 = a_1^{-1}$.

Assume inductively that

$$(d_{r-1}^{-1} \cdots d_1^{-1} c_1 \cdots c_{r-1}, c_r) \in D \text{ and } (d_r^{-1}, d_{r-1}^{-1} \cdots d_1^{-1} c_1 \cdots c_r) \in D$$

and $d_r^{-1} \cdots d_1^{-1} c_1 \cdots c_r = a_r^{-1}$. As (a_r^{-1}, c_{r+1}) and $(a_r^{-1} c_{r+1}, a_{r+1}) \in D$ and $(a_r^{-1} c_{r+1}) a_{r+1} = d_{r+1}$, Lemma 3.2 implies $d_{r+1}^{-1}(a_r^{-1} c_{r+1}) = a_{r+1}^{-1}$. Combined with the inductive hypothesis this shows that the $(r+1)$st version of this hypothesis also holds. Hence the statement of the inductive hypothesis holds for $r = 1,\dots,m$. As $a_m = 1$ we obtain, from the kth version $d_m^{-1} \cdots d_1^{-1} c_1 \cdots c_m = 1$, as required.

Conversely, suppose the conditions given in the lemma hold. Then $(d_1^{-1}, c_1) \in D$ and so we may define $a_1^{-1} = d_1^{-1} c_1$. Two applications of Lemma 3.2 show that $(c_1, a_1) \in D$ and $c_1 a_1 = d_1$. Define $a_0 = 1$ and suppose that $a_1,\dots,a_r$ have been defined such that $(a_{i-1}^{-1}, c_i), (a_{i-1}^{-1} c_i, a_i) \in D$ $a_i^{-1} = d_i^{-1} \cdots d_1^{-1} c_1 \cdots d_i$ and $d_i = a_{i-1}^{-1} d_i a_i$, $i = 1,\dots r$. Then $(d_r^{-1} \cdots d_1^{-1} c_1 \cdots c_r, c_{r+1})$ and $(d_{r+1}^{-1}, d_r^{-1} \cdots d_1^{-1} c_1 \cdots c_{r+1}) \in D$ and we may set $a_{r+1}^{-1} = d_{r+1}^{-1} \cdots d_1^{-1} c_1 \cdots c_{r+1} = d_{r+1}^{-1}(a_r^{-1} c_{r+1})$. Two applications of Lemma 3.2 give $a_r^{-1} c_{r+1} a_{r+1} = d_{r+1}$. Finally we obtain $a_m^{-1} = d_m^{-1} \cdots d_1^{-1} c_1 \cdots c_m = 1$ so $(c_1,\dots,c_m) * (a_1,\dots,a_{m-1})$ is defined and equal to $(d_1,\dots,d_m)$ as required. ☐

Corollary 3.5. *If Q is a subpregroup of a pregroup P then $U(Q)$ is a subgroup of $U(P)$. In particular, if P is an $\mathcal{S}$-pregroupthen $U(P)$ is an $\mathcal{S}$-group.*

Proof. To prove the first statement we need to show that if $\mathbf{a}$ and $\mathbf{b}$ are words over Q then $\mathbf{a} \approx \mathbf{b}$ in Q if and only if $\mathbf{a} \approx \mathbf{b}$ in P. Suppose that $\mathbf{a} \approx \mathbf{b}$ in P. Then using Lemma 3.4 and the definition of $\mathcal{L}^{\text{pre}}$-substructure we have $\mathbf{a} \approx \mathbf{b}$ in Q. As the opposite implication is immediate this proves the first part of the corollary. For the second statement suppose that $K \in \mathcal{S}$ and that K is a subset of a pregroup L, as in the definition above. As $K \subseteq P$ we may assume that $L \subseteq P$ and so $K \subseteq U(L) \subseteq U(P)$. ☐

Theorem 3.6. *Let P_1 and P_2 be $\mathcal{S}$-pregroups. If $P_1 \equiv_\exists P_2$ with respect to $\mathcal{L}_\mathcal{S}^{pre}$ then $U(P_1) \equiv_\exists U(P_2)$ with respect to $\mathcal{L}_\mathcal{S}$.*

Proof. Let $U_i = U(P_i)$ and $D_i = D_{P_i}$, for $i = 1,2$. We shall show that $\mathcal{F}(U_1) \equiv \mathcal{F}(U_2)$ and the theorem will then follow from Proposition 2.2.

Let $F = \{\tilde{u}_1,\dots,\tilde{u}_m\}$ be a finite subset of U_1. For each i choose a representative u_i of $\tilde{u}_i$ and write it as a reduced word $u_i = (c_{i1},\dots,c_{im_i})$ over P_1. Let $S_0 = \cup_{i=1}^m \cup_{j=1}^{m_i} \{c_{ij}\}$ and for all $r \geq 0$ let $S_{r+1} = S_r \cup \{ab : a,b \in S_i \text{ and } (a,b) \in D_1\}$. Let $J = \max\{m_i : i = 1,\dots,m\}$ and define $S = S_{2J}$. As $P_1 \equiv_\exists P_2$ there is, using Proposition 2.2, an $\mathcal{L}_\mathcal{S}^{\text{pre}}$-isomorphism ϕ from S to a subset T of P_2. Note that setting $T_0 = \phi(S_0)$ we may define T_r as we have defined S_r, with T_0 in place of S_0 and P_2 in place of P_1. Then, by definition of isomorphism and by construction of S it follows that $\phi(S_r) = T_r$, for $r = 0,\dots,2J$, so $T = T_{2J}$. Let $\mathbf{c} = (c_1,\dots,c_k)$ be a word over S (i.e. $c_i \in S$, for all i) and let $\phi(c_i) = d_i$. Then $\mathbf{d} = (d_1,\dots,d_k)$ is a word over T and we define a map θ from words over S to words over T by setting $\theta(\mathbf{c}) = \mathbf{d}$. In this case, for all such $\mathbf{c}$, we have $(c_i, c_{i+1}) \in D_1$ if and only if $(d_i, d_{i+1}) \in D_2$, so θ maps reduced words to reduced words.

Now let $\mathbf{p}_1 = (p_{11}, \dots, p_{1m_1})$ and $\mathbf{p}_2 = (p_{21}, \dots, p_{2m_2})$ be words over S_0, with $m_i \le J$. Let $\phi(p_{ij}) = q_{ij}$, and let $\theta(\mathbf{p}_i) = \mathbf{q}_i = (q_{i1}, \dots, q_{im_i})$, $i = 1, 2$. From Lemma 3.4 we have $\mathbf{p}_1 \approx \mathbf{p}_2$ if and only if $m_1 = m_2 = k$, $(p_{2,r-1}^{-1} \cdots p_{2,1}^{-1} p_{1,1} \cdots p_{1,r-1}, p_{1,r})$ and $(p_{2,r}^{-1}, p_{2,r-1}^{-1} \cdots p_{2,1}^{-1} p_{1,1} \cdots p_{1,r-1} p_{1,r})$ belong to D_1, for $r = 1, \dots, k$, and $p_{2,k}^{-1} \cdots p_{2,1}^{-1} p_{1,1} \cdots p_{1,k} = 1$. Since all the elements of P_1 involved in these conditions belong to S, the conditions hold if and only if they hold on replacing p_{ij} with q_{ij}. Hence $\mathbf{p}_1 \approx \mathbf{p}_2$ if and only if $\mathbf{q}_1 \approx \mathbf{q}_2$. Therefore θ induces a map $\tilde{\theta}$ from equivalence classes of reduced words over S_0, of length at most J, to equivalence classes of reduced words over T_0.

Let $\tilde{S}$ and $\tilde{T}$ denote the sets of equivalence classes of reduced words of length at most J, over S_0 and T_0 respectively. To see that the map that $\tilde{\theta}$ is an $\mathcal{L}_{\mathcal{S}}$-morphism from $\tilde{S}$ to $\tilde{T}$ consider a word (not necessarily reduced) $\mathbf{p} = (p_1, \dots, p_k)$ over S_0 of length $k \le J$. Let $q_i = \phi(p_i)$ and let $\theta(\mathbf{p}) = \mathbf{q} = (q_1, \dots, q_k)$. We claim that for r with $0 \le r \le k-1$ there is a sequence of r reductions which we may apply to $\mathbf{p}$, resulting in a word $\mathbf{p}_r$, if and only if there is a corresponding sequence of r reductions which we may apply to $\mathbf{q}$ resulting in a word $\mathbf{q}_r$ such that $\theta(\mathbf{p}_r) = \mathbf{q}_r$. Moreover $\mathbf{p}_r \in S_r$ and $\mathbf{q}_r \in T_r$. This holds trivially for $r = 0$. Suppose that it holds for $0, \dots, r$, for some $0 \le r \le k-2$. Let $\mathbf{p}_r = (p_{r,1}, \dots, p_{r,s})$ and $\mathbf{q}_r = ((q_{r,1}, \dots, q_{r,s})$, with $\mathbf{p}_r \in S_r$ and $\mathbf{q}_r \in T_r$ and $\mathbf{q}_r = \theta(\mathbf{p}_r)$. We may apply a reduction to $\mathbf{p}_r$ if and only if $(p_{r,i}, p_{r,i+1}) \in D_1$, for some i, in which case we may define $\mathbf{p}_{r+1} = (p_{r,1}, \dots, p_{r,i}p_{r,i+1}, \dots p_{r,s})$ and then $\mathbf{p}_{r+1} \in S_{r+1}$. Since ϕ is an $\mathcal{L}_{\mathcal{S}}^{\mathrm{pre}}$-isomorphism this occurs if and only if $(q_{r,i}, q_{r,i+1}) \in D_2$, in which case we may define $\mathbf{q}_{r+1} = (q_{r,1}, \dots, q_{r,i}q_{r,i+1}, \dots q_{r,s})$ and then $\mathbf{q}_{r+1} \in T_{r+1}$. Since $\theta(\mathbf{p}_r) = \mathbf{q}_r$ it follows that $\theta(\mathbf{p}_{r+1}) = \mathbf{q}_{r+1}$ and so the claim holds for all r. Now let $\tilde{p}_1$ and $\tilde{p}_2$ be elements of $\tilde{S}$ and let $\mathbf{p}_1$ and $\mathbf{p}_2$ be reduced words, of length at most J, over S_0 representing $\tilde{p}_1$ and $\tilde{p}_2$, respectively. Suppose that $\mathbf{p}$ is a reduced word (over S) obtained from the concatenation $\mathbf{p}_1\mathbf{p}_2$ by a sequence of reductions. Then, in U_1, we have $\tilde{p}_1\tilde{p}_2 = \tilde{p}$, where $\tilde{p}$ is the equivalence class of $\mathbf{p}$. Let $\mathbf{q}_i = \theta(\mathbf{p}_i)$ and $\mathbf{q} = \theta(\mathbf{p})$. Then, from the above, the concatenation $\mathbf{q}_1\mathbf{q}_2$ reduces to $\mathbf{q}$, which is a reduced word over T. Hence, in U_2, $\tilde{q}_1\tilde{q}_2 = \tilde{q}$, where $\tilde{q}$ is the equivalence class of $\mathbf{q}$. Now, in the case where $\mathbf{p}$ is a word over S_0 we have $\tilde{\theta}(\tilde{p}_1)\tilde{\theta}(\tilde{p}_2) = \tilde{q}_1\tilde{q}_2 = \tilde{q} = \tilde{\theta}(\tilde{p}) = \tilde{\theta}(\tilde{p}_1\tilde{p}_2)$, showing that $\tilde{\theta}$ is an $\mathcal{L}_{\mathcal{S}}$-morphism. Using the result of the first half of this paragraph and the fact that $\tilde{\theta}$ is bijective we can show that $\tilde{\theta}^{-1}$ is also an $\mathcal{L}_{\mathcal{S}}$-morphism. In particular $\tilde{\theta}$ restricted to F is an $\mathcal{L}_{\mathcal{S}}$ isomorphism onto its image. Therefore $\mathcal{F}(U_1) = \mathcal{F}(U_2)$, as required. □

4. Applications

In this section we apply Theorem 3.6 to prove that the universal equivalence of pregroups translates nicely into universal equivalence of free constructions.

4.1. **Free products.** To simplify notation we assume from the outset that we have two groups A and B whose intersection is the identity element. In this case let $P = A \cup B$ and set $D = (A \times A) \cup (B \times B)$. Then P is a pregroup and $U(P) = A * B$.

Proposition 4.1. *Let A_1, B_1, A_2 and B_2 be groups such that $A_1 \cap B_1 = A_2 \cap B_2 = 1$. If $\mathcal{F}(A_1) \equiv \mathcal{F}(A_2)$ and $\mathcal{F}(B_1) \equiv \mathcal{F}(B_2)$ then $A_1 * B_1$ is existentially equivalent to $A_2 * B_2$.*

Proof. Let $P_1 = A_1 \cup B_1$ and $P_2 = A_2 \cup B_2$ be two pregroups as above. Let S be a finite subset of P_1 in the language $\mathcal{L}^{\text{pre}}$. Then $S = (S \cap A_1) \cup (S \cap B_1) = S_{A_1} \cup S_{B_1}$. Let S'_{A_2} and S'_{B_2} be two finite subsets of A_2 and B_2 in the language of groups $\mathcal{L}$, isomorphic to S_{A_1} and S_{B_1}, respectively. Then $S' = S'_{A_2} \cup S'_{B_2}$ is a subset of P_2 isomorphic to S in the language $\mathcal{L}^{\text{pre}}$. By Proposition 2.2, $P_1 \equiv_\exists P_2$ in the language $\mathcal{L}^{\text{pre}}$, and by Theorem 3.6 $A_1 * B_1 \equiv_\exists A_2 * B_2$ in the language $\mathcal{L}$. □

4.2. **Free Products with Amalgamation.** Again it simplifies notation to assume that A and B are C-groups which intersect in the designated copy of the subgroup C, where $C \neq 1$. In this case let $P = A \cup B$ and set $D = (A \times A) \cup (B \times B)$. Then P is a C-pregroup and $U(P) = A *_C B$.

Proposition 4.2. *Let A_1, B_1, A_2 and B_2 be C-groups such that $A_1 \cap B_1 = A_2 \cap B_2 = C$. If $\mathcal{F}_{\mathcal{L}_C}(A_1) \equiv \mathcal{F}_{\mathcal{L}_C}(A_2)$ and $\mathcal{F}_{\mathcal{L}_C}(B_1) \equiv \mathcal{F}_{\mathcal{L}_C}(B_2)$ then the group $A_1 *_{C_1} B_1$ is existentially equivalent to $A_2 *_{C_2} B_2$ in the language $\mathcal{L}_C$ and, a fortiori, in the language $\mathcal{L}$.*

Proof. Let $P_1 = A_1 \cup B_1$ and $P_2 = A_2 \cup B_2$, be two C-pregroups as above. Let S be a finite subset of P_1 in the language $\mathcal{L}_C^{\text{pre}}$. Let $S_{A_1} = S \cap A_1$ and $S_{B_1} = S \cap B_1$ so $S = S_{A_1} \cup S_{B_1}$. Let S'_{A_2} and S'_{B_2} be two finite subsets of A_2 and B_2 in the language of C-groups $\mathcal{L}_C$, isomorphic to S_{A_1} and S_{B_1}, respectively. Then $S' = S'_{A_2} \cup S'_{B_2}$ is a subset of P_2 isomorphic to S in the language $\mathcal{L}_C^{\text{pre}}$. By Proposition 2.2, $P_1 \equiv_\exists P_2$ in the language $\mathcal{L}_C^{\text{pre}}$, and by Theorem 3.6 $A_1 *_C B_1 \equiv_\exists A_2 *_C B_2$ in the language $\mathcal{L}_C$. □

4.3. **HNN-Extensions.** Given a group G and an isomorphism $\theta : C_1 \to C_2$, where C_1 and C_2 are subgroups of G, let t be a symbol not in G and

$$P_0 = G \cup t^{-1}G \cup Gt \cup t^{-1}Gt. \tag{2}$$

Let P be the set of equivalence classes of the equivalence relation generated by $t^{-1}ht = \theta(h)$, for all $h \in C_1$. Set

$$D = \bigcup_{\varepsilon_0,\varepsilon_1,\varepsilon_2=0,1} t^{-\varepsilon_0}Gt^{\varepsilon_1} \times t^{-\varepsilon_1}Gt^{\varepsilon_2} \subseteq P \times P.$$

(The equivalence relation means that $(p, \theta(c))$ and $(\theta(c), p)$ belong to D for all $c \in C_1$ and $p \in P$.) Then P and D constitute a pregroup and it can be verified that C_1 and C_2 embed in P. Hence P is an $\mathcal{S}$-pregroup, where $\mathcal{S} = \{C_1, C_2\}$. Moreover $U(P)$ is the HNN-extension $\langle G, t | t^{-1}ct = \theta(c), c \in C_1\rangle$, which is an $\mathcal{S}$-group (with constants C_1 and $\theta(C_1) = C_2$).

Proposition 4.3. *Let A_1 and A_2 be $\mathcal{S}$-groups, where $\mathcal{S} = \{C_1, C_2\}$ and $\theta : C_1 \to C_2$ is an isomorphism. If $\mathcal{F}_{\mathcal{L}_\mathcal{S}}(A_1) \equiv \mathcal{F}_{\mathcal{L}_\mathcal{S}}(A_2)$ then the group $G_1 = \langle A_1, t \mid t^{-1}ct = \theta(c), c \in C_1\rangle$ is existentially equivalent, in the language $\mathcal{L}_\mathcal{S}$ and in the language $\mathcal{L}$, to the group $G_2 = \langle A_2, t \mid t^{-1}ct = c, c \in C_1\rangle$.*

Proof. Let P_1 and P_2 be the two $\mathcal{S}$-pregroups corresponding to A_1 and A_2, respectively, as defined above, and let $P_{1,0}$ and $P_{2,0}$ be the underlying sets, as in (2). Let S be a finite subset of P_1 in the language $\mathcal{L}_\mathcal{S}^{\text{pre}}$. Let $\hat{S} \subseteq P_{1,0}$ be the union of all the equivalence classes of elements of S. Then $\hat{S}$ is a disjoint union of 4 sets, $S_1 = \hat{S} \cap A_1$, $S_2 = \hat{S} \cap t^{-1}A_1$, $S_3 = \hat{S} \cap A_1 t$ and $S_4 = \hat{S} \cap t^{-1}A_1 t$. To obtain corresponding sets in A_1 define $T_1 = S_1$, $T_2 = tS_2$, $T_3 = S_3 t^{-1}$ and $T_4 = tS_4 t^{-1}$. By hypothesis there exist subsets $T'_i \subseteq A_2$, such that $T_i \cong_{\mathcal{L}_\mathcal{S}} T'_i$, for $i = 1, \ldots, 4$.

Set $S_1' = T_1'$, $S_2' = t^{-1}T_2'$, $S_3' = T_3't$ and $S_4' = t^{-1}T_4't$. Define $\hat{S}' = S_1' \cup S_2' \cup S_3' \cup S_4'$. The $\mathcal{L}_{\mathcal{S}}$-isomorphisms between the T_i's and the T_i''s induce a bijection from $\hat{S}$ to $\hat{S}'$ and by construction this isomorphism factors through the equivalence relations on $P_{1,0}$ and $P_{2,0}$ to give an $\mathcal{L}_{\mathcal{S}}^{\mathrm{pre}}$-isomorphism between S and the quotient S' of $\hat{S}'$ in P_2. Applying Proposition 2.2 and Theorem 3.6, G_1 is universally equivalent to G_2 in the language $\mathcal{L}_{\mathcal{S}}$ (and consequently in the language $\mathcal{L}$). □

References

[1] G. Baumslag, A. Myasnikov, and V.N. Remeslennikov, *Algebraic geometry over groups. I. Algebraic sets and ideal theory,* J. Algebra **219** (1999), 16-79.

[2] D. Marker *Model Theory: An Introduction.* Springer, 2002.

[3] J. Stallings, *Group Theory and Three-Dimensional Manifolds.* Yale Mathematical Monographs **4**, Yale Univ. Press, 1971.

School of Mathematics and Statistics, University of Newcastle-upon-Tyne, Newcastle-Upon-Tyne NE1 7RU, United Kingdom

E-mail address: a.duncan@ncl.ac.uk

Department of Mathematics and Statistics, McGill University, 805 Sherbrooke St. West, Montreal, Quebec H3A 2K6, Canada

E-mail address: kazachkov@math.mcgill.ca

Institute of Mathematics (Russian Academy of Science), 13 Pevtsova St., Omsk, 644099, Russia

E-mail address: remesl@iitam.omsk.net.ru